Resit exam Analysis on Manifolds

June 30, 2017

This exam consists of **three** assignments. You get 10 points for free.

Assignment 1. (10+10+10=30 pt.)
We identify the space $M(2,\mathbb{R})$ of 2×2-matrices with $\mathbb{R}^4$, by associating the matrix

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

with the point $(x_{11}, x_{12}, x_{21}, x_{22}) \in \mathbb{R}^4$.

1. Show that the set

   $$SL(2,\mathbb{R}) = \{X \in M(2,\mathbb{R}) \mid \det X = 1\}$$

   is a 3-dimensional C^∞-submanifold of $M(2,\mathbb{R})$. Here $\det X$ denotes the determinant of X. (Hint: translate every statement in terms of subsets of $\mathbb{R}^4$.)

2. Show that the tangent space $T_E\,SL(2,\mathbb{R})$ at the identity matrix

   $$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   is equal to $\{A \in M(2,\mathbb{R}) \mid \operatorname{Tr} A = 0\}$. Here $\operatorname{Tr} A$ is the *trace* of the matrix A, i.e., the sum of the diagonal entries of A.

3. The map $f : SL(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is given by $f(X) = X^{-1}$. This map is differentiable (you don't have to prove this). Show that

   $$d_E f : T_E\,SL(2,\mathbb{R}) \to T_E\,SL(2,\mathbb{R})$$

   is given by $d_E f(A) = -A$.

Assignment 2. (10+10+10=30 pt.)
Let X be a vector field on $\mathbb{R}^n$, f a smooth function on $\mathbb{R}^n$, and ω an n-form on $\mathbb{R}^n$.
1. Prove that $df \wedge \iota_X\omega = X(f)\,\omega$.

Recall that $\iota_X\omega$ is the $(n-1)$–form given by $\iota_X\omega(X_1,\dots,X_{n-1}) = \omega(X, X_1,\dots,X_{n-1})$, and that for a vector field $X = \sum_{i=1}^n a_i E_i$ (or, equivalently, $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$) the function $X(f)$ is defined by $X(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}$.

2. Let σ be a 1-form and let η be an $(n-1)$-form on $\mathbb{R}^n$. Define

$$\omega : \underbrace{\mathcal{X}(\mathbb{R}^n) \times \cdots \times \mathcal{X}(\mathbb{R}^n)}_{n \text{ times}} \to C^\infty(\mathbb{R}^n)$$

by

$$\omega(Y_1,\dots,Y_n) = \sum_{i=1}^{n} (-1)^{i-1} \sigma(Y_i)\,\eta(Y_1,\dots,\widehat{Y_i},\dots,Y_n).$$

Prove that ω is an n-form on $\mathbb{R}^n$.

3. In the notation of part 2, prove that $\omega = \sigma \wedge \eta$.

Assignment 3 on next page

Assignment 3. (7+5+6+6+6=30 pt.)
Note: the solution of this assignment is likely to be shorter than its statement.

Let the map $f : M \to \mathbb{R}^n$ be a C^∞-embedding of a compact orientable m-dimensional C^∞-manifold M *without boundary*, with $m \leq n$ (so, in particular, $f(M)$ is an m-dimensional C^∞-submanifold of $\mathbb{R}^n$).

1. If ω is an m-form on $\mathbb{R}^n$ with $\int_M f^*\omega \neq 0$, then ω is not exact on $\mathbb{R}^n$. Prove this.

Let M be the two-dimensional C^∞-submanifold of $\mathbb{R}^4$ given by

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 = 1, x_3^2 + x_4^2 = 1\},$$

and let $i : M \to \mathbb{R}^4$ be the inclusion map (the canonical embedding). You don't have to prove that M is a C^∞-submanifold. Let η_1 and η_2 be the one-forms on $\mathbb{R}^4$ given by

$$\eta_1 = -x_2\, dx_1 + x_1\, dx_2 \quad \text{and} \quad \eta_2 = -x_4\, dx_3 + x_3\, dx_4.$$

2. Let $p = (p_1, p_2, p_3, p_4) \in M$. Prove that a basis of T_pM is $\{v_1, v_2\}$, where

   $$v_1 = -p_2 e_1 + p_1 e_2 \quad \text{and} \quad v_2 = -p_4 e_3 + p_3 e_4.$$

   As usual, e_j is the j-th standard basis vector of $\mathbb{R}^4$.

3. Prove that $i^*(\eta_1 \wedge \eta_2)$ is a nowhere-zero two-form on M.

   (Hint: prove that $i^*(\eta_1 \wedge \eta_2)_p(v_1, v_2) \neq 0$, for $p \in M$ and $v_1, v_2 \in T_pM$ as in part 2 of this assignment. Use that $di_p : T_pM \to T_p\mathbb{R}^4$ is the inclusion map.)

4. Prove that the two-form $\eta_1 \wedge \eta_2$ on $\mathbb{R}^4$ is not exact.

   (Hint: consider using part 1 and part 3.)

5. Prove that the one-forms $i^*\eta_1$ and $i^*\eta_2$ on M are not exact.

   (Hint: let $\mathbb{S}_1 = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 = 1\}$ be the unit circle in $\mathbb{R}^2$, and let $f_1 : \mathbb{S}^1 \to M$ be the embedding given by $f_1(u, v) = (u, v, 1, 0)$. Apply part 1 of this exercise to show that $i^*\eta_1$ is not exact. Follow a similar approach to show that $i^*\eta_2$ is not exact.)

Solutions

Assignment 1.
1. Let $F : \mathbb{R}^4 \to \mathbb{R}$ be the C^∞-map defined by $F(x_{11}, x_{12}, x_{21}, x_{22}) = x_{11}x_{22} - x_{12}x_{21}$. Then $d_X F$ has matrix $\begin{pmatrix} x_{22} & -x_{21} & -x_{12} & x_{11} \end{pmatrix}$, which has maximal rank (namely, one) for $X \in SL(2, \mathbb{R})$. Therefore, $SL(2, \mathbb{R})$ is a 3-dimensional C^∞-submanifold of $\mathbb{R}^4$.

2. Use that $T_E\, SL(2, \mathbb{R}) = \ker d_E F$, and that $X \in \ker d_E F$ iff $x_{11} + x_{22} = 0$ iff $\operatorname{Tr} X = 0$.

3. Let $A \in T_E\, SL(2, \mathbb{R})$, and let $X : \mathbb{R} \to SL(2, \mathbb{R})$ be a C^∞-curve with $X(0) = E$ and $X'(0) = A$. Then $d_E f(A) = Y'(0)$, where $Y(t) = f(X(t)) = X(t)^{-1}$.

Since $X(t) \cdot Y(t) = E$, we see that $X'(0) \cdot Y(0) + X(0) \cdot Y'(0) = 0$ (the zero matrix), so, using $X(0) = Y(0) = E$ we get $Y'(0) = -A$.

Assignment 2. Let $X = \sum_{i=1}^n a_i \dfrac{\partial}{\partial x_i}$ and let $\omega = g\, dx_1 \wedge \cdots \wedge dx_n$, then

$$\iota_X \omega = g\,\Big(\sum_{i=1}^n (-1)^{i-1} a_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots dx_n\,\Big).$$

Since $df = \sum_{i=1}^n \dfrac{\partial f}{\partial x_i}\, dx_i$, we get

$$\begin{aligned}
df \wedge \iota_X \omega &= g\,\Big(\sum_{j=1}^n \frac{\partial f}{\partial x_j}\, dx_j\Big) \wedge \Big(\sum_{i=1}^n (-1)^{i-1} a_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\,\Big) \\
&= g\,\Big(\sum_{i=1}^n (-1)^{i-1} a_i \frac{\partial f}{\partial x_i}\, dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\Big) \\
&= \Big(\sum_{i=1}^n a_i \frac{\partial f}{\partial x_i}\Big)\, g\, dx_1 \wedge \cdots \wedge dx_n \\
&= X(f)\, \omega.
\end{aligned}$$

2. The map ω is multilinear over $C^\infty(\mathbb{R}^n)$. To see that it is antisymmetric, let us swap Y_1 and Y_2. Note that

$$\begin{aligned}
\omega(Y_1, Y_2, Y_3, \ldots, Y_n) &= \sigma(Y_1)\, \eta(Y_2, Y_3, \ldots, Y_n) - \sigma(Y_2)\, \eta(Y_1, Y_3, \ldots, Y_n) \\
&\qquad + \sigma(Y_3)\, \eta(Y_1, Y_2, \widehat{Y_3}, \ldots, Y_n) \ldots \\
&= -\sigma(Y_2)\, \eta(Y_1, Y_3, \ldots, Y_n) + \sigma(Y_1)\, \eta(Y_2, Y_3, \ldots, Y_n) \\
&\qquad - \sigma(Y_3)\, \eta(Y_2, Y_1, \widehat{Y_3}, \ldots, Y_n) \ldots \\
&= -\omega(Y_2, Y_1, Y_3, \ldots, Y_n).
\end{aligned}$$

One similarly proves that swapping any other pair of arguments introduces a minus-sign. Therefore, ω is an n-form.

3. Since $\omega = \omega(\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n})\, dx_1 \wedge \cdots \wedge dx_n$, we only have to prove the desired identity for $Y_i = \dfrac{\partial}{\partial x_i}$. Since

$$\sigma = \sum_{i=1}^n \sigma_i\, dx_i, \quad \text{with} \quad \sigma_i = \sigma\Big(\frac{\partial}{\partial x_i}\Big),$$

and

$$\eta = \sum_{i=1}^{n} \eta_i \, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n, \text{ with } \eta_i = \eta(\frac{\partial}{\partial x_1}, \ldots, \widehat{\frac{\partial}{\partial x_i}}, \ldots, \frac{\partial}{\partial x_n}),$$

we have, according to the definition of the exterior product,

$$\sigma \wedge \eta = \sum_{i=1}^{n} (-1)^{i-1} \, \sigma_i \eta_i \, dx_1 \wedge \ldots \wedge dx_n.$$

Therefore,

$$\begin{aligned}(\sigma \wedge \eta)(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) &= \sum_{i=1}^{n} (-1)^{i-1} \, \sigma_i \eta_i \\ &= \sum_{i=1}^{n} (-1)^{i-1} \, \sigma(\frac{\partial}{\partial x_i}) \eta(\frac{\partial}{\partial x_1}, \ldots, \widehat{\frac{\partial}{\partial x_i}}, \ldots, \frac{\partial}{\partial x_n}) \\ &= \omega(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}).\end{aligned}$$

Hence $\sigma \wedge \eta = \omega$.

Assignment 3.

1. Assume $\omega = d\varphi$. Use Stokes: $0 \neq \int_M f^*\omega = \int_M f^* d\varphi = \int_M d(f^*\varphi) = \int_{\partial M} f^*\varphi = 0$, since $\partial M = \emptyset$. This contradiction shows that ω is not exact.

2. Note that $M = F^{-1}(0)$, with $F : \mathbb{R}^4 \to \mathbb{R}^2$ given by $F(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2 - 1, x_3^2 + x_4^2 - 1)$. It is easy to see that dF_p has matrix $\begin{pmatrix} 2p_1 & 2p_2 & 2p_3 & 2p_4 \end{pmatrix}$, which has rank 2 at points $p \in M$. Hence $T_pM = \ker dF_p$. Since v_1 and v_2 are independent, and $dF_p(v_1) = dF_p(v_2) = 0$, the claim follows.

3. A straightforward computation shows that

$$(i^*\eta_k)_p(v_l) = (\eta_k)_p(di_p(v_l)) = (\eta_k)_p(v_l) = \delta_{kl}.$$

Hence $(i^*(\eta_1 \wedge \eta_2))_p(v_1, v_2) = ((i^*\eta_1)_p \wedge (i^*\eta_2)_p)(v_1, v_2) = 1$, by the definition of wedge product of two one-forms. Since for $p \in M$, the system $\{v_1, v_2\}$ is a basis of T_pM, the claim follows.

4. From part 3 we conclude $\int_M i^*(\eta_1 \wedge \eta_2) \neq 0$. Now use part 1 to conclude that $\eta_1 \wedge \eta_2$ is not exact on $\mathbb{R}^4$. (Remark: a completely different approach consists of proving that $d(\eta_1 \wedge \eta_2) \neq 0$, by a straightforward though tedious calculation.)

5. Let $f_1 : \mathbb{S}^1 \to M$ be the given embedding, then the pull-back of $i^*\eta_1$ under f_1 is the one-form on $\mathbb{S}_1$ given by $f_1^* i^*\eta_1 = -v\,du + u\,dv$. This one-form is nowhere zero on $\mathbb{S}^1$, so $\int_{\mathbb{S}^1} f_1^* i^*\eta_1 \neq 0$. Use part 1 to conclude that $i^*\eta_1$ is not exact. To prove that $i^*\eta_2$ is not exact use the embedding $f_2 : \mathbb{S}^1 \to M$ given by $f(u, v) = (1, 0, u, v)$ and argue similarly.